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An upper bound for $\|A^{-1}\|_\infty$ of strictly diagonally dominant M-matrices

Guang-Hui Cheng *, Ting-Zhu Huang [1]

School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, PR China

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Abstract

Let A be a real strictly diagonally dominant M-matrix. We give a sharp upper bound for $\|A^{-1}\|_\infty$. Furthermore, the lower bound of the smallest eigenvalue $q(A)$ is established.

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1. Introduction

A square $n \times n$ matrix A is called a nonsingular M-matrix if there exists an $n \times n$ nonnegative matrix P such that

$$A = sI - P, \tag{1}$$

where I is the identity matrix, and $s > \rho(P)$, the spectral radius of nonnegative matrix P. If A is an M-matrix, there exists a positive eigenvalue of A equal to $\frac{1}{\rho(A^{-1})}$, where $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}. We denote this eigenvalue by $q(A)$, then $q(A) = s - \rho(P)$ is also the minimum of the real parts of the eigenvalues of A. See [1–3], for example, for further discussion of this issue.

* Corresponding author.
E-mail addresses: ghcheng@uestc.edu.cn (G.-H. Cheng), tzhuang@uestc.edu.cn (T.-Z. Huang).
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Shivakumar et al. [4] gave the following results on two-sided bounds related to weakly chained diagonally dominant M-matrices.

Theorem 1.1 [4]. *Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant M-matrix, let $A^{-1} = (\alpha_{ij})$, and let $q = q(A)$, $N = \{1, 2, \ldots, n\}$. Then*

$$q \leqslant \min\{a_{ii} : i \in N\}, \tag{2}$$

$$q \leqslant \max\left\{\sum_{j\in N} a_{ij} : i \in N\right\}, \tag{3}$$

$$q \geqslant \min\left\{\sum_{j\in N} a_{ij} : i \in N\right\}, \tag{4}$$

$$\frac{1}{M} \leqslant q \leqslant \frac{1}{m}, \tag{5}$$

where

$$M = \max_{i\in N} \sum_{j\in N} \alpha_{ij} = \|A^{-1}\|_\infty \quad \textit{and} \quad m = \min_{i\in N} \sum_{j\in N} \alpha_{ij}.$$

Varah [7] gave the following result.

Theorem 1.2 [7]. *If $A = a_{ij} \in \mathbb{R}^{n\times n}$ is a strictly diagonally dominant matrix, then*

$$\|A^{-1}\|_\infty \leqslant \frac{1}{\min_i \left\{|a_{ii}| - \sum_{j\neq i} |a_{ij}|\right\}}, \quad i \in N. \tag{6}$$

Remark 1.1. If the diagonal dominance of A is weak, i.e., $\min_i\{|a_{ii}| - \sum_{j\neq i} |a_{m_{ij}}|\}$ is small, then using Theorem 1.2 in estimating $\|A^{-1}\|_\infty$, the bound is may yield a large value.

In this paper, the upper bound of $\|A^{-1}\|_\infty$ is improved, furthermore, by using (5), a new lower bound of $q(A)$ is obtained.

The paper is organized as follows. In Section 2, we present notations and some preliminary results. The main results are driven for M-matrices in Section 3.

2. Notations and preliminaries

In the following, we need the following definitions and results. They will be useful in the following proofs. For convenience, for any positive integer n, N denotes the set $\{1, 2, \ldots, n\}$ throughout. Let $A = (a_{ij})$ be an $n \times n$ matrix. For any $i \in N$, denote

$$d_i = \frac{1}{|a_{ii}|} \sum_{j\neq i} |a_{ij}|, \quad J(A) = \{i \in N : d_i < 1\},$$

$$u_i = \frac{1}{|a_{ii}|} \sum_{j=i+1}^{n} |a_{ij}|,$$

$$l_k = \max\left\{\frac{\sum_{j\neq i+k-1, k\leqslant j\leqslant n}|a_{i+k-1,j}|}{|a_{i+k-1,i+k-1}|} : i = 1, 2, \ldots, n-k+1\right\}.$$

Definition 2.1 [4]. $A \in \mathbb{R}^{n\times n}$ is weakly chained diagonally dominant if for all $i \in N$, $d_i \leqslant 1$ and $J(A) \neq \emptyset$, and for all $i \in N$, $i \notin J(A)$, there exist indices $i_1, i_2, \ldots, i_k$ in N with $a_{i_r, i_{r+1}} \neq 0$, $0 \leqslant r \leqslant k-1$, where $i_0 = i$ and $i_k \in J(A)$.

Definition 2.2 [8]. $A \in \mathbb{R}^{n\times n}$ is an L-matrix if for all $i, j \in N$ with $i \neq j$, $a_{ij} \leqslant 0$ and $a_{ii} > 0$.

Definition 2.3 [8]. Let $A \in \mathbb{R}^{n\times n}$. A is strictly diagonally dominant if $J(A) = N$.

Lemma 2.1 [5]. *A weakly chained diagonally dominant L-matrix is a nonsingular M-matrix.*

We will denote by $A^{(n_1,n_2)}$ as the principal submatrix of A formed from all rows and all columns with indices between n_1 and n_2 inclusively; e.g., $A^{(2,n)}$ is the submatrix of A obtained by deleting the first row and the first column of A.

Lemma 2.2 [5]. *Let A be an $n \times n$ weakly chained diagonally dominant M-matrix. Then $B = A^{(2,n)}$ is an $(n-1)\times(n-1)$ weakly chained diagonally dominant M-matrix, i.e., $B^{-1} = (\beta_{ij})$ exists and $\beta_{ij} \geqslant 0$, $i, j = 2, 3, \ldots, n$.*

Lemma 2.3 [5]. *Let $A = (a_{ij})$ be a weakly chained diagonally dominant M-matrix and $A^{-1} = (\alpha_{ij})$. Then, for $i \neq j$,*

$$\alpha_{ij} \leqslant d_i\alpha_{jj} \leqslant \alpha_{jj}. \tag{7}$$

Lemma 2.4 [6]. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix, and $A^{-1} = (\alpha_{ij})$. Then*

$$\alpha_{ii} \geqslant \left(a_{ii} - \sum_{k\neq i}\frac{a_{ik}a_{ki}}{a_{kk}}\right)^{-1} \geqslant a_{ii}^{-1}, \quad i \in N. \tag{8}$$

3. Upper bounds for $\|A^{-1}\|_\infty$

In this section, we give upper bounds for the inverse of a strictly diagonally dominant M-matrix.

Theorem 3.1. *Let $A = (a_{ij})$ be a weakly chained diagonally dominant M-matrix, $B = A^{(2,n)}$, $A^{-1} = (\alpha_{ij})_{i,j=1}^n$, and $B^{-1} = (\beta_{ij})_{i,j=2}^n$. Then, for $i, j = 2, \ldots, n$,*

$$\alpha_{11} = \frac{1}{\Delta}, \tag{9}$$

$$\alpha_{i1} = \frac{1}{\Delta}\sum_{k=2}^{n}\beta_{ik}(-a_{k1}), \tag{10}$$

$$\alpha_{1j} = \frac{1}{\Delta}\sum_{k=2}^{n}\beta_{kj}(-a_{1k}), \tag{11}$$

$$\alpha_{ij} = \beta_{ij} + \alpha_{1j}\sum_{k=2}^{n}\beta_{ik}(-a_{k1}), \tag{12}$$

where

$$\Delta = a_{11} - \sum_{k=2}^{n} a_{1k}\left(\sum_{i=2}^{n}\beta_{ki}a_{i1}\right) > 0. \tag{13}$$

Furthermore, *if* $J(A) = N$, *we have*

$$\Delta \geqslant a_{11}(1 - d_1 l_1) \geqslant a_{11}(1 - d_1). \tag{14}$$

Proof. Eqs. (9)–(13) were proved in [4]. For $k, j \geqslant 2$,

$$\sum_{i=2}^{n}\beta_{ki}a_{ij} = (B^{-1}B)_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

From (13), using Lemmas 2.3 and 2.4 we have

$$\begin{aligned}
\Delta &= a_{11} - \sum_{k=2}^{n} a_{1k}\left(\sum_{i=2}^{n}\beta_{ki}a_{i1}\right) \\
&= a_{11} - \sum_{k=2}^{n} a_{1k}\left[\sum_{i=2}^{n}\beta_{ki}a_{ii}(1 - d_i) - \sum_{j=2}^{n}\sum_{i=2}^{n}\beta_{ki}a_{ij}\right] \\
&= a_{11} - \sum_{k=2}^{n} a_{1k}\left[\sum_{i=2}^{n}\beta_{ki}a_{ii}(1 - d_i) - 1\right] \\
&= \sum_{k=1}^{n} a_{1k} + \sum_{k=2}^{n}(-a_{1k})\sum_{i=2}^{n}\beta_{ki}a_{ii}(1 - d_i) \\
&\geqslant \sum_{k=1}^{n} a_{1k} + (1 - l_1)\sum_{k=2}^{n}(-a_{1k})\sum_{i=2}^{n}\beta_{ki}a_{ii} \\
&\geqslant \sum_{k=1}^{n} a_{1k} + (1 - l_1)\sum_{k=2}^{n}(-a_{1k})\sum_{i=2}^{n}|\beta_{ki}a_{ik}| \\
&= \sum_{k=1}^{n} a_{1k} + (1 - l_1)\sum_{k=2}^{n}(-a_{1k})(2\beta_{kk}a_{kk} - 1) \\
&\geqslant \sum_{k=1}^{n} a_{1k} + (1 - l_1)\sum_{k=2}^{n}(-a_{1k}) \\
&= a_{11}(1 - d_1 l_1) \\
&\geqslant a_{11}(1 - d_1) \\
&\geqslant 0. \qquad \square
\end{aligned} \tag{15}$$

Corollary 3.2. *If $A = (a_{ij})$ is an $n \times n$ row strictly diagonally dominant M-matrix, then $\Delta \geqslant a_{11}(1 - d_1 l_1) > a_{11}(1 - d_1) > 0$.*

Proof. Since $A = (a_{ij})$ is an $n \times n$ strictly diagonally dominant M-matrix, $0 < d_i < 1, i \in N$ and $0 \leqslant l_1 < 1$, i.e., $a_{11}(1 - d_1 l_1) > a_{11}(1 - d_1) > 0$, the conclusion follows. □

Theorem 3.3. *Let $A = (a_{ij})$ be an $n \times n$ strictly diagonally dominant M-matrix and $B = A^{(2,n)}$, $A^{-1} = (\alpha_{ij})_{i,j=1}^n$, and $B^{-1} = (\beta_{ij})_{i,j=2}^n$. Then*

$$\|A^{-1}\|_\infty \leqslant (1 - d_1)\alpha_{11} + \frac{d_1}{a_{11}(1 - d_1 l_1)} + \frac{1}{1 - d_1 l_1}\|B^{-1}\|_\infty, \tag{16}$$

and

$$\|A^{-1}\|_\infty \leqslant \frac{1}{a_{11}(1 - d_1 l_1)} + \left(1 + \frac{d_1}{1 - d_1 l_1}\right)\|B^{-1}\|_\infty. \tag{17}$$

Proof. Let

$$r_i = \sum_{j=1}^n \alpha_{ij}, \quad M_A = \|A^{-1}\|_\infty, \quad M_B = \|B^{-1}\|_\infty.$$

Then

$$M_A = \max\{r_i : i \in N\} \quad \text{and} \quad M_B = \max\left\{\sum_{j=2}^n \beta_{ij} : 2 \leqslant i \leqslant n\right\}.$$

By Theorem 3.1 and Corollary 3.2,

$$\begin{aligned} r_1 &= \alpha_{11} + \sum_{j=2}^n \alpha_{1j} \\ &= \frac{1}{\Delta} + \frac{1}{\Delta}\sum_{k=2}^n(-a_{1k})\sum_{j=2}^n \beta_{kj} \\ &\leqslant \frac{1}{\Delta} + \frac{1}{\Delta}\sum_{k=2}^n(-a_{1k})M_B \\ &\leqslant \frac{1 + a_{11}d_1 M_B}{a_{11}(1 - d_1 l_1)}. \end{aligned} \tag{18}$$

Let $2 \leqslant i \leqslant n$. Then, using (7) in (9) and (10),

$$\sum_{k=2}^n \beta_{ik}(-a_{k1}) \leqslant d_i < 1.$$

From (12), with $2 \leqslant j \leqslant n$, we have

$$\alpha_{ij} \leqslant \beta_{ij} + \alpha_{1j}d_i < \beta_{ij} + \alpha_{1j}. \tag{19}$$

Thus, for $2 \leqslant i \leqslant n$, we obtain

$$r_i = \alpha_{i1} + \sum_{j=2}^n \alpha_{ij}$$

$$\begin{aligned}&\leqslant \alpha_{11} + \sum_{j=2}^{n}(\beta_{ij} + \alpha_{1j}d_i)\\&\leqslant (1 - l_1)\alpha_{11} + l_1 r_1 + M_B,\end{aligned} \tag{20}$$

or, we can obtain

$$r_i \leqslant r_1 + M_B. \tag{21}$$

According to (20) and (21), we have

$$M_A \leqslant (1 - l_1)\alpha_{11} + l_1 r_1 + M_B \quad \text{and} \quad M_A \leqslant r_1 + M_B,$$

respectively. Using (18), the results follow. □

Theorem 3.4. *Let $A = (a_{ij})$ be an $n \times n$ strictly diagonally dominant M-matrix. Then*

$$\|A^{-1}\|_\infty \leqslant \frac{1}{a_{11}(1 - u_1 l_1)} + \sum_{i=2}^{n}\left[\frac{1}{a_{ii}(1 - u_i l_i)}\prod_{j=1}^{i-1}\left(1 + \frac{u_j}{1 - u_j l_j}\right)\right]. \tag{22}$$

Proof. Apply induction with respect to k to $A^{(k,n)}$, using (17). □

Remark 3.1. Using (16), (17) and (22) in (5), we can obtain new bounds of the smallest eigenvalue $q(A)$.

Theorem 3.5. *Let $A = (a_{ij})$ be an $n \times n$ strictly diagonally dominant M-matrix. Then the bound in* (22) *is sharper than that in Theorem* 3.3 *in* [4], *i.e.*,

$$\frac{1}{a_{11}(1 - u_1 l_1)} + \sum_{i=2}^{n}\left[\frac{1}{a_{ii}(1 - u_i l_i)}\prod_{j=1}^{i-1}\left(1 + \frac{u_j}{1 - u_j l_j}\right)\right] < \sum_{i=1}^{n}\left[a_{ii}\prod_{j=1}^{i}(1 - u_i)\right]^{-1}. \tag{23}$$

Proof. Since A is a strictly diagonally dominant matrix, $0 \leqslant l_k < 1$ for all k. Consequently, we have $\frac{1}{a_{11}(1-d_1 l_1)} < \frac{1}{a_{11}(1-d_1)}$ and $\left(1 + \frac{d_1}{1-d_1 l_1}\right)\|B^{-1}\|_\infty < \frac{1}{1-d_1}\|B^{-1}\|_\infty$, i.e.,

$$\frac{1}{a_{11}(1 - d_1 l_1)} + \left(1 + \frac{d_1}{1 - d_1 l_1}\right)\|B^{-1}\|_\infty < \frac{1}{a_{11}(1 - d_1)} + \frac{1}{1 - d_1}\|B^{-1}\|_\infty. \tag{24}$$

The inequality (24) shows that the bound in (17) is better than that in Lemma 3.2 in [4], further, the bound in (22) is sharper than that in Theorem 3.3 in [4]. □

In the following, we give a numerical example to illustrate the results obtained in Section 3.

Example 3.1. Let

$$A = \begin{bmatrix} 1 & 0 & -0.2 \\ -0.8 & 1 & -0.1 \\ -0.9 & 0 & 1 \end{bmatrix}.$$

By direct calculations with Matlab 7.0, we have

$$A^{-1} = \begin{bmatrix} 1.2195 & 0 & 0.2439 \\ 1.0854 & 1.0000 & 0.3171 \\ 1.0976 & 0 & 1.2195 \end{bmatrix}, \quad \|A^{-1}\|_\infty = 2.4025.$$

We have

$$\|A^{-1}\|_\infty \leqslant 10 \text{ (by Theorem 1.2)},$$
$$\|A^{-1}\|_\infty \leqslant 4.0278 \text{ (by Theorem 3.3 in [4])},$$
$$\|A^{-1}\|_\infty \leqslant 3.8455 \text{ (by Theorem 3.4)},$$

respectively. It is obviously that the bound of Theorem 3.4 is sharper. Furthermore, we can use Theorem 3.4 and (5) in estimating $q(A)$.

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